Classical and Multivalued Logics: Foundations and Computational Applications (LOGFAC)

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Abstract

In this project we are working on classical and multivalued propositional logic as a foundation of computer science. We are looking at the foundations of multivalued logic based on t-norms, and the proof theory of both classical and multivalued logics. Moreover, we are applying this knowledge to automated deduction, hardware verification, and SAT-based problem solving. The topics we are studying are: (1) logic and algebraic foundations of multivalued logic underlying fuzzy logics; (2) application of these logics to the study of approximate reasoning in intelligent systems; (3) study of classical deduction systems from the point of view of computational complexity; (4) extension to multivalued logic of the results obtained in classical proof theory; (5) theoretical foundation of automated deduction; and (6) design and implementation of algorithms for classical and multivalued SAT.

Keywords: Classical Boolean logic, multivalued logic, automated deduction, approximate reasoning, satisfiability algorithms.

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1 Objectives of the Project

The objectives of the project are the following ones:

- **Logical and algebraic study of t-norm-based residuated multivalued logics.**
  
  Our objectives are both in the algebraic semantics for the family of t-norm based many-valued logics as well as in the corresponding varieties and the logical interpretation of the results obtained. We are also interested in studying these logics in relation to residuated and substructural logics which, starting from different motivations, are at the roots of the most general t-norm based many-valued logic, namely MTL, the logic of left-continuous t-norms and their residua.

- **Foundations of approximate reasoning in the framework of t-norm-based residuated multivalued logics.**
  
  One objective is, following preliminary results already obtained, to explore the use of some t-norm based many-valued logics as a logical framework where to define modal theories corresponding to different uncertainty logics, like probabilistic reasoning, possibilistic logic or a belief function logic. This approach can also be used to get a formalization of similarity-based reasoning. Another objective is to deepen into the formalization of different aspects of fuzzy logic in a broad sense, like fuzzy rules or modifiers, which are not explicitly dealt in the above systems of many-valued logic. These aspects are very important for defining practical systems of approximate reasoning.

- **Proofs of lower bounds for propositional proof systems.**
  
  The relationship between the complexity classes P and NP is fundamental in computer science. It is known that NP $\neq$ Co-NP if and only if for every propositional proof system there is a class of tautologies that requires superpolynomial size proofs (respect to the size of the tautology) in that system. Therefore if we were able to prove for every propositional proof system such lower bounds, then NP $\neq$ Co-NP, and because the class P is closed under complement, P $\neq$ NP.

  For some quite strong propositional proof systems classes of tautologies that require long proofs are known. We want to continue proving such results for stronger propositional proof systems. Also we want to study the comparative strength between systems, to be able to obtain a better classification.

- **Theoretical analysis of the limits of automated theorem proving.**
  
  We expect that the complexity classes P, NP and Co-NP are different. Therefore, as a conclusion of what we have said in the previous item, we expect that for every proof system there will be classes of tautologies that will require superpolynomial size proofs. This is really bad news for propositional automated theorem provers. This is because for some tautologies, only to be able to write down their proofs we will need exponential time.

  As a consequence we propose a different attack. The question is: what happens with tautologies that have short proofs in some proof system? Can we come up with an algorithm that finds these quickly? Bonet-Pitassi-Raz proposed the following definition.
A proof system $P$ is *automatizable* if there is an algorithm that given a formula, it finds a proof of it in the system $P$ in time polynomial in the size of the shortest $P$-proof of the formula. Bonet-Pitassi-Raz showed that some strong systems don’t have this property. We want to investigate if weaker systems like Resolution or Cutting Planes are automatizable.

Also we want to study other robust notions related to automatizability, like feasible interpolation or less restrictive versions of automatizability, and see which proof systems have them.

- **Relationship between logical circuits and proof theory.**
  A proof line for some propositional proof system corresponds to a type of circuit. This relationship has helped show that some formulas require long proofs in some propositional proof systems using lower bounds on the corresponding circuits. Also it has helped give small proofs in some other cases. It would be interesting to study this relationship better, not only for theoretical purposes, but also for applications. Good algorithms for automated deduction can help to do circuit verification.

- **SAT-based problem solving.**
  Motivated by the success of the generic problem solving approach that consists of translating a given problem into Boolean propositional satisfiability (SAT), solving it with a fast SAT solver and mapping the solution back into the original problem, our goal is to investigate the benefits one can gain by using multivalued SAT solvers and encodings. To this end, we will study logical and complexity aspects of different multivalued clausal form formalisms, design and implement competitive multivalued SAT solvers, and apply the multivalued SAT-based problem solving approach to solve real-world problems.

- **Phase transition phenomenon in multivalued SAT problems.**
  The study of search behavior of random SAT formulas has provided tremendous insights into the hardness nature of combinatorial problems, beyond the worst-case notion of NP-completeness. In particular, such studies have uncovered an interesting phase transition behavior between an area in which most instances are solvable and one in which most of the instances are unsolvable. The critically constrained area, where the hardest instances occur, coincides with the phase transition. Our goal is to identify similar phase transition phenomena in multivalued SAT problems and provide a theoretical explanation of such phenomena.

## 2 Results of the Project

The results obtained for each one of the objectives of the project are the next ones:

- **Logical and algebraic study of t-norm-based residuated multivalued logics.**
  In the framework of the Basic Fuzzy logic BL, the logic defined by Hájek to cope with the logic of continuous t-norms and its residuum, we have studied and axiomatized some subvarieties of the variety of the corresponding algebras (BL-algebras). In [28] and [29] we have studied the varieties $\mathcal{V}(C)$ generated by a BL-chain $C$ belonging to a special
class (called regular BL-chains) that contains the standard BL-chains (BL-chains defined on \([0,1]\) by a continuous t-norm and its residuum). We have proved that each such subvariety is finitely axiomatizable and that the family of subvarieties \(V(C)\) for \(C\) regular is countable. Moreover we have provided algorithms to check embedding between subvarieties of the family and to find the set of equations \(Eq(C)\) defining \(V(C)\). This last result is specially important since, in particular, we obtain in this way all the axiomatic extensions of BL logic that are standard complete with respect to any standard BL-chain. In other words, we have obtained an axiomatization of any fuzzy logic in a narrow sense relative to a continuous t-norm and its residuum.

On the other hand, Esteva and Godo had recently defined the logic MTL, generalization of BL, proved to be the logic of left continuous t-norms (the most general t-norms having residuum) and their residua. In this framework we have studied several issues. Namely, in [30] we have addressed completeness problems of some axiomatic extensions of MTL, in [27] the falsehood-free fragment of MTL and BL, in [32] the axiomatic extensions of the Nilpotent Minimum logic, an extension in turn of MTL, in [25] the problem of definability of connectives, and finally in [26] we have studied the class of t-norm based logics in the framework of residuated and substructural logics.

It is also worth mentioning a related work, [21], where completeness results and a proof theory for MTL are generalized to the axiomatic extension of MTL adding the \(n\)-contraction axiom \((x^n = x^{n+1})\). This work has been done in the framework of the Integrated Action HU2001-0030 between IIIA and the Logic Group of the Technical University of Vienna (Austria).

- **Foundations of approximate reasoning in the framework of t-norm-based residuated multivalued logics.**

  We have studied three logical aspects of approximate reasoning. In [39] we have introduced a new class of fuzzy closure operators called implicative closure operators, which generalize most of the notions of fuzzy closure operators already introduced by different authors. We show that implicative closure operators capture most usual consequence relations used in Approximate Reasoning, in particular those which are based on the notion of similarity. In [35, 36, 22] we have studied the use of similarity relations in systems of implicative fuzzy rules for solving problems of coverage and inconsistency. We have shown how the notion of similarity can address these problems in a natural way and moreover in [23] we have shown how the similarity relation can be learned from the fuzzy system itself. Finally, in [1, 2] we have also contributed to further developments in the topic of an extended possibilistic logic framework which allows to deal with fuzzy object constants and which incorporates a fuzzy unification mechanism for such constants.

- **Proofs of lower bounds for propositional proof systems.**

  We have proved lower bounds for various complexity measures related to propositional proof systems: size, space and rank.

  First of all we will discuss the size lower bounds, which consists in proving that some classes of tautologies require superpolynomial size proofs in some proof systems. In [13, 14, 24] we study the proof systems Res\((k)\), which are extensions of Resolution that allow clauses with conjunctions of up to \(k\) literals. These systems are interesting also from
the point of view of automated deduction, because they are stronger than Resolution.
Given that general proof search algorithms produce tree-like refutations (each line is used
at most once), we have also studied the complexity of tree-like Res($k$). In [24] we show
that Resolution is exponentially stronger than tree-like Res($k$). This means that there is a
class of tautologies that has polynomial size proofs in Resolution but require exponential
size tree-like Res($k$) size. Also in [24] we show exponential separations between Res($k$)
and Res($k + 1$) for all $k$.

In [14] we formulate a simple form of consistency and we show which proof systems
among the Res($k$) ones prove their own consistency or the consistency of other systems
in polynomial size. Also we see that Resolution requires exponential size proofs of the
formula that expresses its own consistency.

The group has also worked with the measure of space. In [24] we prove space lower
bounds for tautologies in the systems Res($k$). In [17] a relationship between the space
and the width is presented. The width of a refutation is the number of literals in the
clause with the highest number of literals. [17] show that the width can be characterized
in terms of game theory for expressivity in infinitary logic. As a consequence of this we
were able to show that the space is never less than the width in Resolution.

Finally we introduced the rank measure for the proof system Cutting Planes. This system
translates sets of unsatisfiable clauses to linear inequalities, and uses certain inference
rules to work with such inequalities. The sets of inequalities are polytopes without
interior integer points. In [16] [20] we use linear programming techniques to study the
Cutting Planes system, using the rank measure. The rank is the number of iterations
of the cut rule until we obtain the integer hull of the polytope. In [16] we show that
the rank measure corresponds to the depth of Cutting Planes refutations. In [16] [20] we
prove lower bounds on the rank of the main used classes of tautologies.

• **Theoretical analysis of the limits of automated theorem proving.**

Our studies of the systems Res($k$) and tree-like Res($k$) are important to automated theo-
rem proving. The conclusion that we obtained is that it is worthwhile to explore versions
of proof search algorithms based on DLL and others that branch out on conjunctions of
literals.

On the other hand our study on the rank of Cutting Planes proof system gives us ideas on
where to look for algorithms for cutting planes. In [16] we show that rank $d$ implies tree-
like size $O(n^d)$. So it would be interesting to come up with an algorithm that obtained
refutations, based on allowing the rank to be increased step by step.

• **Relationship between logical circuits and proof theory.**

The close relationship between the theory of Boolean circuits and the theory of proposi-
tional proof complexity has been observed by a number of researchers more than once.
Nonetheless, the established connections were empirical only, lacking a theoretical back-
ground. In the articles [13] [12] we achieve a precise provable relationship between the
size of proofs in resolution of a propositional formula, and inexpressivity results in logical
formalisms such as Datalog. These results, novel in both areas, suggest that the rela-
tionship between proof complexity and classical complexity is better understood through
descriptive complexity theory than through Boolean circuit complexity. The discovery of
this connection has been a surprise in both research areas, to the extent that it provided
the tools to solve some problems that were left open in the literature for some time. In
the article [17] we prove, using these techniques, that the resolution space is always big-
ger than the resolution width. The exact relationship between both measures remained
unknown despite being well-known in the areas of resolution and automatic deduction.

• **SAT-based problem solving.**

  We have defined resolution calculi for a wide range of many-valued clausal forms and
  have analyzed the complexity of their Horn-SAT and 2-SAT problems [5, 10]. Then, we
  have developed a multivalued local search SAT solver inspired by the Boolean SAT solver
  WalkSAT [9, 11], and two multivalued complete solvers inspired by zChaff [8] and Satz [7].
  Finally, we have shown that multivalued SAT-based problem solving outperforms Boolean
  SAT-based problem solving in a considerable number of problems [9, 11, 8, 7]. This is
due to the fact that multivalued SAT preserves and exploits better the structure of the
problem, provides more powerful propagation techniques and gives rise to more compact
encodings. We are now studying the competitiveness of multivalued SAT for hardware
verification.

• **Phase transition phenomenon in multivalued SAT problems.**

  We have identified phase transition phenomena for several multivalued SAT problems,
  and have studied empirically the interface between P and NP for such problems [6]. The
  instances of the hard region of the phase transition have been used to experimentally
evaluate the multivalued SAT solvers developed. We are now investigating how to provide
a theoretical explanation of such phenomena.

3 **Indicators of Results**

7 Ph.D. students are or have been working on the project: C. Ansótegui (UdL), J. Argelich
(UdL), A. Atserias (UPC), J.L. Esteban (UPC), J. Larrubia (UdL), C. Noguera(IIIA-CSIC)
Bounded Propositional Proofs”, at UPC in 2002. C. Ansótegui and J.L. Esteban will submit
their Ph.D. thesis within the next two months.

The results of the project have been published in books, conference proceedings and inter-
national journals. The list of publications appears in the References section. This section only
contains the publications of the project.

In addition to the publications presented in the conferences some members of the research
team have given invited talks, have organized conferences and have been members of pro-
gramme committees. F. Esteva gave invited talks in the Congreso Español de Lógica y Tec-
nología Fuzzy (ESTYLF’02) and the Symposium on Fuzzy Logic associated to the International
Congress of Logic, Methodology and Philosophy of Science. L. Godo gave an invited talk in
the First International Workshop on Knowledge Representation and Approximate Reasoning
(Olsztyn, Poland) and taught a course on Soft Computing at the Third International Summer
School on Reasoning under Partial Knowledge (REASONPARK-03) in Perugia, Italia. J. Levy
organized and chaired the 17th International Workshop on Unification (UNIF’03) in Valencia.
A. Atserias gave invited talks in the 13th Annual Conference of the European Association for Computer Science in Karpacz, in the Workshop on the Satisfiability Problem in Dagstuhl, and in the Workshop on Typical Case Complexity and Phase Transition in Ottawa.

The topics of the project have given rise to international collaborations. The IIIA-CSIC team has a project (acción integrada) with the group of Matthias Baaz of the Technical University of Wien, and a project with P. Hájek of the Czech Academy of Sciences. The UPC team has had a project (acción integrada HA 2000-41) with the group in Ulm (Germany) led by Jacobo Torán. The UdL team has a project with the group of Bart Selman and Carla Gomes from Cornell University (USA), funded by the Air Force Office for Scientific Research.

Apart from these projects there are ongoing collaborations with Chu Min Li (Université de Picardie), R. Hähnle (Chalmers University), Toniann Pitassi (University of Toronto), Pavel Pudlák (Czech Academy of Sciences), D. Mundici (Univ. Florence), F. Montagna (Univ. Siena), B. Gerla (Univ. Salerno), etc.

A remarkable effort has been devoted to the development of software. During this year, there has been implemented a local search multivalued SAT solver and two complete multivalued SAT solvers, one that incorporates look-ahead heuristics and one that incorporates look-back heuristics.

References


